18.02A Recitation — Wednesday, Oct. 24, 2018

Quick Review

- Project a vector $\mathbf{v}$ onto another vector $\mathbf{w}$

$$\left(\mathbf{v} \cdot \frac{\mathbf{w}}{||\mathbf{w}||}\right) \frac{\mathbf{w}}{||\mathbf{w}||}$$

- Determinant of a 3-by-3 matrix
- Cross product $\mathbf{v} \times \mathbf{w}$ (works in 3D, the product is another 3D-vector)
  1. Geometrically: $||\mathbf{v} \times \mathbf{w}||$ is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$; $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ and the direction is determined by *right-hand rule.*
  2. Algebraically,

$$\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Practice problems:

1. Project the vector $\langle 1, 2, 3 \rangle$ onto $\langle 1, 0, 1 \rangle$.

2. Compute the following determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

3. Compute the cross product

$$\langle 2, -1, 3\rangle \times \langle -4, 3, -5\rangle.$$

Solution.

$$\begin{aligned}\langle 2, -1, 3\rangle \times \langle -4, 3, -5\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -4 & 3 & -5 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 3 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix}\mathbf{k} \\ &= -4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

□

4. (Optional) Let $\mathbf{v} = \langle 2, -1, 3\rangle$ and $\mathbf{w} = \langle -4, 3, -5\rangle$. The two lines generated by $\mathbf{v}$ and $\mathbf{w}$ will determine a plane P.

   Suppose $\mathbf{A} = \langle 1, 2, 3\rangle$. Find two vectors $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$ with $\mathbf{B}$ parallel to the plane P and $\mathbf{C}$ perpendicular to the plane P.

   Solution. Let $\mathbf{q} = \mathbf{v} \times \mathbf{w}$. By the previous question, we already know that $\mathbf{q} = \langle -4, -2, 2\rangle$. Furthermore, by definition of cross product, we know $\mathbf{q}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and hence also to the plane P.

   Since $\mathbf{C}$ is perpendicular to the plane P, $\mathbf{C}$ should be parallel to $\mathbf{q}$. Similarly, since $\mathbf{B}$ is parallel to the plane P, $\mathbf{B}$ should be perpendicular to $\mathbf{q}$.

   So, now the original question converts to the following problem: knowing $\mathbf{A}$ and $\mathbf{q}$, find two vectors $\mathbf{B}$ and $\mathbf{C}$ with $\mathbf{B}$ perpendicular to $\mathbf{q}$ and $\mathbf{C}$ parallel to $\mathbf{q}$.

   Hence, $\mathbf{C}$ is nothing but the projection of $\mathbf{A}$ onto $\mathbf{q}$:

$$\mathbf{C} = \left(\frac{\mathbf{A} \cdot \mathbf{q}}{||\mathbf{q}||}\right)\frac{\mathbf{q}}{||\mathbf{q}||} = \langle \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}\rangle.$$

   Now, since $\mathbf{A} = \mathbf{B} + \mathbf{C}$, we have

$$\mathbf{B} = \mathbf{A} - \mathbf{C} = \langle \frac{2}{3}, \frac{11}{6}, \frac{19}{6}\rangle.$$

□