18.02A Recitation — Wednesday, Nov. 14, 2018

Quick Review

- Gradient.
  - For two-variable function $z = f(x, y)$,
  $$\vec{\nabla} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle;$$
  for three-variable function $w = f(x, y, z)$,
  $$\vec{\nabla} f(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\rangle.$$
  - Geometrically, direction of $\vec{\nabla} f$ points to the direction of the steepest ascent; $||\vec{\nabla} f||$ is the rate of increase in that direction. (BTW, what is the direction of the fastest descent?)
  - More geometric meaning: gradient vector is perpendicular to the level curve (for two-variable function) or the level surface (for three-variable function). That is:
    * For two-variable function $z = f(x, y)$, if $f(x_0, y_0) = c$, then $\vec{\nabla} f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = c$;
    * For three-variable function $w = f(x, y, z)$, if $f(x_0, y_0, z_0) = c$, then $\vec{\nabla} f(x_0, y_0, z_0)$ is perpendicular to the level surface $f(x, y, z) = c$.
- Directional derivative. Let $\mathbf{u}$ be a **unit vector**. The directional derivative of f is given by
  $$D_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u}.$$
  Warning: Be very careful here! **u must be a unit vector!!!** If not, you need to normalize the vector (by dividing it by its length).
- Min/Max problem. To find the min/max problem for a two-variable function over a region R,
  1. Find critical point(s): solve $\vec{\nabla} f(x, y) = 0$.
  2. Study the value of f over the boundary of R. This will usually reduce f into a one-variable function.
  3. Find the min/max point by comparing the value of f at critical point(s) and boundary of R.

Practice problems:

1. (PSet 3 Question. Skip it if you had no problem with it.) You are climbing a mountain whose height is given by $z = f(x, y) = 1000 - 2x^2 - 3y^2$.

   a) Find the directional derivative of $f(x, y)$ in the radial direction at any point $P = (x_0, y_0)$, where $(x_0, y_0) \neq (0, 0)$. (*The radial direction is the unit vector with the same direction as the vector* $\overrightarrow{OP}$.)

b) When you are at the point $(1, 1, 995)$, in what direction in the (x, y)-plane should you initially move in order to ascend as rapidly as possible? Give your answer in the form of a unit vector with the direction you want.

c) Suppose you move on a path whose projection to the (x, y)-plane is given by $(g(t), h(t))$ and always moves in the direction of steepest ascent (meaning in the same direction as the unit vector describing steepest ascent). Show that $h'(t)/g'(t) = 3/2 \cdot h(t)/g(t)$.

2. Find the equation of the tangent plane to

$$xyz + x^2 - 2y^2 + z^3 = 14,$$

at the point $(5, -2, 3)$.

Hint: view it as a level surface of a three-variable function.

3. Find the global min/max of the function $f(x, y) = x^2 + y^2 - 2x$ on the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$.